On periods of Herman rings and relevant poles

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Abstract

Possible periods of Herman rings are studied for general meromorphic functions with at least one omitted value. A pole is called H-relevant for a Herman ring H of such a function f if it is surrounded by some Herman ring of the cycle containing H. In this article, a lower bound on the period p of a Herman ring H is found in terms of the number, h of H-relevant poles. More precisely, it is shown that $p \geq \frac{h(h+1)}{2}$ whenever $f^j(H)$, for some j, surrounds a pole as well as the set of all omitted values of f. It is proved that $p \geq \frac{h(h+3)}{2}$ in the other situation. Sufficient conditions are found under which equalities hold. It is also proved that if at least one of the omitted value is contained in an invariant or a two periodic Fatou component then the function does not have any Herman ring.

Keyword: Omitted values, Herman rings and Transcendental meromorphic functions.

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1 Introduction

Let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ be a transcendental meromorphic function such that it has either at least two poles or exactly one pole which is not an omitted value. For such functions, there are infinitely many points whose iterated forward image is infinity, the only essential singularity of the function. This is the reason why these are called general meromorphic functions. The class of all such functions is denoted by M in the literature [2]. A family of meromorphic functions defined on a domain is called normal if each sequence taken from the family has a subsequence that converges uniformly on every compact subset of the domain. The limit is allowed to be infinity. The Fatou set of f is the set of all points in a neighborhood of which the family of functions $\{f^n\}_{n>0}$ is well defined and normal. For general meromorphic functions, normality is in fact redundant. More precisely, the Fatou set of a general meromorphic function is the set of all points where f^n is defined for all n [2]. Its complement is the Julia set.

The Fatou set is open by definition. A maximal connected subset of the Fatou set is called a Fatou component. For a Fatou component U and a natural number k, let U_k denote the Fatou component containing $f^k(U)$. A Fatou component U is called p-periodic if p is the smallest natural number satisfying $U_p = U$. We say U is invariant if $p = 1$. A Fatou component U is called completely invariant if it is forward invariant ($f(U) \subseteq U$) as well as backward invariant ($f^{-1}(U) \subseteq U$). If U is not periodic but U_k is periodic for some natural number k, then U is called pre-periodic. If a Fatou component is neither periodic nor pre-periodic, then it is called a wandering domain. The connectivity of a periodic Fatou component is known to be $1,\ 2$ or ∞ [2]. The sequence of iterates f^n has finitely many limit functions on a periodic Fatou component. Depending on whether such limit functions are constants or not, a periodic Fatou component can be an attracting domain, a parabolic domain, a Baker domain, a Siegel disk, or a Herman ring. The last two possibilities arise

precisely when the limit functions are non-constant. This article is mainly concerned with Herman rings.

A p-periodic Fatou component H is called a Herman ring if there exists an analytic homeomorphism $\phi : H \to \{z : 1 < |z| < r\}$ such that f^p is conformally conjugate to an irrational rotation. In other words, $\phi(f^p(\phi^{-1}(z))) = e^{i2\pi\alpha}z$ for some irrational number α and for all $z, 1 < |z| < r$. Clearly, a Herman ring is multiply connected.

Using the Maximum Modulus Principle, it can be shown that transcendental entire functions can not have any Herman ring. More precisely, every multiply connected Fatou component of these functions is wandering [2]. Thus multiply connected Fatou components are well understood for transcendental entire functions. The situation can be more complicated for transcendental meromorphic functions. This is a case when, for example, there is a Herman ring. A meromorphic function of finite order can have at most finitely many Herman rings whereas, there are transcendental meromorphic functions having infinitely many Herman rings. This is proved by Zheng in [10]. Dominguez et al. showed that, for a given $N > 0$, there exists an $f \in M$ with exactly N poles and N invariant Herman rings [4].

Non-existence of Herman rings seems to allow certain kind of simplicity in the dynamics of a function. A Herman ring is always doubly connected giving rise to a disconnected Julia set. In other words, a connected Julia set ensures the non-existence of Herman rings. Baranski and co-authors proved that transcendental meromorphic functions arising as Newton maps of entire functions have connected Julia sets, and hence have no Herman ring [1]. Another class of functions, namely those general meromorphic functions omitting at least one value are studied by Nayak and co-authors [6, 7]. A number of sufficient conditions guaranteeing the non-existence of Herman ring are provided by the authors. In particular, following are proved. If all the poles of such a function are multiple, then it has no Herman ring. Functions with a single pole or with

at least two poles, one of which is an omitted value, have no Herman ring. Herman rings of period one or two do not exist. Examples of functions which has no Herman ring are also provided in [3]. In view of all these, following conjecture can be made.

Conjecture 1.1. *If a general meromorphic function omits at least one point in the plane then it does not have any Herman ring.*

This is the motivation for the current work.

A value $z_0 \in \widehat{\mathbb{C}}$ is said to be an *omitted value* of a function f if $f(z) \neq z_0$ for any $z \in \mathbb{C}$. Let O_f denote the set of all omitted values of f. Note that O_f consists of at most two points, and is a subset of the plane whenever $f \in M$.

Let M_o be the set of all functions in M having at least one omitted value. All functions considered in this article belongs to M_o.

Nayak has proved that, if $f \in M_o$ then f has no Herman ring of period 1 or 2 [6]. The proof contains a detailed analysis of the possible arrangements of Herman rings in the plane relative to each other. We say a set is surrounded by a Herman ring H if the set is contained in the bounded component of the complement of H. The locations of the omitted value(s) and poles surrounded by Herman rings have also been key to a number of useful observations. Later, these observations are used to show that there can not be more than one p-cycles of Herman rings for $p = 3, 4$. These ideas are developed and used in this article to prove a lower bound for periods of Herman rings and non-existence of the same under certain situation.

Given a Herman ring H, a pole w is said to be H-relevant if some ring H_i of the cycle containing H surrounds w. A lower bound on the period p of a Herman ring H is found in terms of the number, h of H-relevant poles. More precisely, it is shown that $p \geq \frac{h(h+1)}{2}$ whenever the basic nest (See Section 2 for definition) surrounds a pole. Less technically, this condition is equivalent to the statement that H_j, for some j, surrounds a pole as well as O_f. It is

proved that $p \geq \frac{h(h+3)}{2}$ in the other situation, when the basic nest does not surround any pole. This is the statement of Theorem 3.5.

A ring is called outermost if it is not surrounded by any other ring (Section 2 can be seen for definition) of the cycle. Similarly, the innermost ring H_1 with respect to the set O_f is a ring which surrounds O_f but does not surround any other ring. It follows from Lemma 2.1 (which originally appeared in [7]) that H_{1+k}, the ring containing $f^k(H_1)$ surrounds a pole of f for some k. The smallest such natural number is what we refer as the length of the basic chain. It is seen that the length of the basic chain is at least h (Lemma 3.2). When it is the least possible or one bigger than that, we are able to prove equality in Theorem 3.5 under some additional condition. Theorem 3.6 proves the following. If each ring surrounding a pole is the outermost ring of the nest then it is proved that (i) $p = \frac{h(h+1)}{2}$ when the length of the basic chain is h, and (ii) $p = \frac{h(h+3)}{2}$ when the length of the basic chain is $h+1$ and the basic nest does not surround any pole. It is worth noting that the assumption of (i) ensures that the basic nest surrounds a pole. The condition that each ring surrounding a pole is the outermost ring of the nest is satisfied whenever the period of a Herman ring is 3 ([3]).

A Herman ring is never completely invariant. It is well known that if there is a completely invariant Fatou component then every other Fatou component is simply connected, and hence there cannot be any Herman ring. Theorem 3.8 proves that if the omitted value is contained in a periodic Fatou component U of f and f has a Herman ring H then the number of H-relevant poles is strictly less than the period of U. This leads to non-existence of Herman rings whenever U is invariant or 2-periodic. This is shown in Theorem 3.8 giving a new condition under which Conjecture 1.1 is true.

Section 2 discusses all the preliminary ideas and results required for proving the results. All the results are stated and proved in Section 3.

We reserve the notation f for functions in M_o throughout this article. The

set of all omitted values of f is denoted by O_f. By a ring, we mean a Herman ring in this article. For a ring H, let $B(H)$ denote the bounded component of the complement of H. We say H surrounds a set A (or a point w) if $A \subset B(H)$ (or if $w \in B(H)$ respectively). For a p-periodic Herman ring H, denote the cycle of H by $\{H_0,\ H_1, \ldots, H_{p-1}\}$, where $H = H_0 = H_p$.

2 Preliminaries

A Jordan curve in a multiply connected Fatou component of a meromorphic function can be considered such that it is not contractible in the Fatou component. Since the backward orbit of ∞ does not intersect the Fatou set, f^n is well defined on such a Jordan curve. The following lemma, proved in [7] analyzes the iterated forward images of such a Jordan curve leading to useful conclusions.

Lemma 2.1. *Let $f \in M$ and V be a multiply connected Fatou component of f. Also let γ be a non-contractible closed curve in V (that means $B(\gamma) \cap \mathcal{J}(f) \neq \phi$). Then there exists an $n \in \mathbb{N} \cup \{0\}$ and a closed curve $\gamma_n \subset f^n(\gamma)$ in V_n such that $B(\gamma_n)$ contains a pole of f. Further, if $O_f \neq \phi$, then $O_f \subset B(\gamma_{n+1})$ for some closed curve γ_{n+1} contained in $f(\gamma_n)$.*

That a multiply connected Fatou component corresponds to a pole follows from the above lemma. A Herman ring is doubly connected. Above lemma applied to a Herman ring gives rise to the following.

Remark 2.1. *Let H be a p-periodic Herman ring of f and $\phi : H \to \{z : 1 < |z| < r\}$ be the analytic homeomorphism such that $\phi(f^p(\phi^{-1}(z))) = e^{i2\pi\alpha}z$ for some irrational number α and for all $z, 1 < |z| < r$. If γ is the pre-image of a circle $\{z : |z| = r'\}$ for $1 < r' < r$ under ϕ then γ is an f^p-invariant and non-contractible Jordan curve in V and the set $\{\gamma_n := f^n(\gamma) : n > 0\}$ is a finite set of f^p-invariant Jordan curves. Further, there is a j such that γ_j surrounds a pole of f.*

Recall that for a p-periodic Herman ring H, the cycle of H is denoted by $\{H_0,\ H_1, \ldots, H_{p-1}\}$, where $H = H_0 = H_p$. All the definitions given and used in this article are with respect to H. The next two definitions were introduced in [6].

Definition 2.2. (H-relevant pole)
Given a Herman ring H, a pole w is said to be H-relevant if some ring H_i of the cycle containing H surrounds w.

It is clear from Remark 2.1 that an f^p-invariant Jordan curve surrounds a pole. Since the curve is in a ring, the existence of at least one H-relevant pole is assured. A refinement of this statement is implicit in the following theorem, proved in [6].

Theorem 2.3. *If $f \in M_o$ has only one pole, then f has no Herman ring.*

It follows from above theorem that the number of H-relevant poles is at least 2 for every function $f \in M_o$.

The position of rings relative to each other is going to play an important role in our investigation.

Definition 2.4. (H-maximal nest)
Given a Herman ring H, a ring H_j is called an H-outermost ring if H_i does not surround H_j for any i, $i \neq j$. Given an outermost ring H_j, the collection of all rings consisting of H_j and all those surrounded by H_j is called an H-maximal nest. We call it simply a nest whenever H is understood from the context.

A nest is a sub-collection of Herman rings from the periodic cycle containing H. Each H_i belongs to exactly one nest. By saying a nest surrounds a point (or a set), we mean the outermost ring of the nest surrounds the point (or the set respectively). This is also true whenever any other ring belonging to the nest surrounds the point or the set. It is important to note that a nest

surrounds at most one pole. This follows from Lemma 2.4, [6], which is stated below.

Lemma 2.2. *If H is a Herman ring of $f \in M_o$, then $f : B(H) \to \widehat{\mathbb{C}}$ is one-one.*

Above lemma also gives that, if all the poles of a function belonging to M_o are multiple, then it has no Herman ring (Corollary 2.6, [6]). Note that f is not one-one in the plane even though it is so in $B(H)$ for every Herman ring.

If a ring H_i surrounds a pole then consider a non-contractible f^p-invariant Jordan curve γ_i in H_i. Now, γ_i surrounds the pole and it follows from the last part of Lemma 2.1 that $f(\gamma_i)$ surround O_f. In other words, H_{i+1} surrounds O_f. The nest containing H_{i+1} is too important to have a name.

Definition 2.5. (Basic nest) *Given a Herman ring H, the $H-$maximal nest surrounding the set of all omitted values of f is called the basic nest of H. A nest different from the basic nest is called non-basic.*

Here is a useful remark.

Remark 2.6. *If a ring of a cycle of Herman rings surrounds a pole then its image surrounds O_f and therefore is in the basic nest.*

We need the idea of innermost rings for making some new definitions.

Definition 2.7. (Innermost ring with respect to a set) *Given a Herman ring H, we say a ring H_j is innermost with respect to a set S if H_j surrounds S but not H_i for any $i, i \neq j$.*

Existence of more than one innermost ring in a nest cannot be ruled out.

Definition 2.8. (Basic chain and Basic rings) *Given a Herman ring H, the ordered set of rings $\{H_1, H_2, H_3, \ldots, H_n\}$ is called the basic chain, where H_1 is the innermost ring with respect to O_f and n is the smallest natural number such that H_n surrounds a pole. Each ring H_i, $1 \leq i \leq n$ is said to be a basic ring of H.*

Now onwards, we reserve H_1 to denote the innermost ring with respect to O_f. The ring H_1 does not surround any pole by Remark 2.10 of [3]. Thus the number of basic rings is at least 2. Here is a useful remark following from the periodicity of Herman rings.

Remark 2.9. *For each ring H' in a non-basic nest, there is a j such that $H_{1+j} = H'$.*

Instead of starting from the innermost ring with respect to O_f, one can start from any ring H_r surrounding O_f but not any pole and look at the smallest m for which H_{r+m} surrounds a pole. This gives rise to the following definition.

Definition 2.10. (Chain) *The ordered set of rings $C = \{H_r, H_{r+1}, \ldots, H_{r+m}\}$ is called a chain if H_r is a ring surrounding O_f but not any pole, and m is the smallest natural number such that H_{r+m} surrounds a pole. The number of rings in a chain C is called its length, and is denoted by $|C|$.*

Note that the basic chain is the unique chain whose first ring is H_1. It is of course a basic ring. Further, the first ring of every chain belongs to the basic nest and the length of every chain is at least two. It is important to note that the last ring of a chain C surrounds a pole, say w. We say C corresponds to w. Though two chains corresponding to the same pole cannot be ruled out, chains corresponding to different poles are important for our purpose.

Definition 2.11. (Independent chains) *Two chains are called independent if they correspond to two different poles.*

Here are two basic observations on the length of chains.

Lemma 2.3.
1. *The length of every chain is less than or equal to that of the basic chain.*
2. *If C_i and C_j are two independent chains then their lengths are different.*

Proof. 1. Let $\{H_{r+1}, H_{r+2}, \cdots, H_{r+n}\}$ be a chain different from the basic chain. Then H_{r+i} does not surround any pole for $i = 1, 2, \cdots, n-1$. Clearly H_{r+1} surrounds H_1, the innermost ring with respect to O_f. Since H_1 is the first ring of the basic chain, it follows from the Maximum Modulus Principle that H_{r+i} surrounds H_i for each $i = 1, 2, \cdots, n$. The pole corresponding to the basic chain is surrounded by either H_n or H_k for some $k > n$. This gives that the length of every chain is less than or equal to that of the basic chain.

2. Let C_i and C_j be two independent chains. By definition of independent chain, the poles w_i and w_j corresponding to C_i and C_j respectively are different. Let H_i and H_j be the initial rings of C_i and C_j respectively. Then both of these surround O_f and hence, either $H_i \subseteq B(H_j)$ or $H_j \subseteq B(H_i)$. Without loss of generality, let $H_i \subseteq B(H_j)$. If the length of C_i and C_j is the same, say l, then H_{j+l} surrounds H_{i+l} which gives that both are contained in the same nest. Also H_{i+l} and H_{j+l} surround the poles w_i and w_j respectively. But each nest surrounds at most one pole (by Lemma 2.2) giving that $w_i = w_j$ which is a contradiction. This proves that the length of C_i is different from that of C_j.

□

3 Results and their proofs

The basic chain, introduced in the previous section is going to play a key role in the proofs. To start with, we make an observation on how it restricts the number of nests in a cycle of Herman rings.

Lemma 3.1. *Let H be a p-periodic Herman ring of f. Then the number of nests in the cycle of H is at most the length of the basic chain.*

Proof. We first show that each nest contains at least one basic ring. This is

clearly true for the basic nest. Now let N be a non-basic nest. Recall that H_1 is the innermost ring with respect to O_f and it does not surround any pole. The ring H_1 is not in N and one of its iterated forward image is in N by Remark 2.9. Let n be the smallest natural number such that H_n is in N. The ring H_{n-1}, the periodic pre-image of H_n, does not surround any pole by Remark 2.6.

If H_k surrounds a pole for some $k,\ 1 < k < n$ then consider the largest such k and denote it by k^*. As observed in the previous paragraph, $k^* \neq n-1$. Therefore $2 \leq k^* \leq n-2$. It follows from Remark 2.6 that H_{k^*+1} is in the basic nest. Further, none of $H_{k^*+1}, H_{k^*+2}, \ldots, H_{k^*+n-k^*-1} = H_{n-1}$ surrounds any pole by the choice of k^*. Since H_1 is the innermost ring with respect to O_f, either H_{k^*+1} surrounds H_1 or is equal to H_1. It follows from Lemma 2.2 that H_{k^*+j} surrounds or is equal to H_j for each $j,\ 1 \leq j \leq n-k^*$. Further, the map $f^{n-k^*-1} : B(H_{k^*+1}) \to B(H_n)$ is conformal. This gives that H_{n-k^*} is a ring in the nest N. However, this contradicts our earlier assumption that n is the smallest natural number such that H_n is in N. This proves that H_k does not surround any pole for any $k,\ 1 < k < n$ and hence H_n is a basic ring.

Since two different nests cannot contain the same basic ring, the number of nests is at most the number of basic rings. The proof is completed by noting that the number of basic rings is nothing but the length of the basic chain.

$\square$

The number of H-relevant poles is at least two by Lemma 2.11 of [3]. An upper bound for this number can be obtained using the previous lemma.

Lemma 3.2. *Let H be a Herman ring of f. Then the number of H- relevant poles is at most the length of the basic chain.*

Proof. By definition, the length of the basic chain is at least two. If the number of H-relevant poles is two then there is nothing to prove. Lemma 3.1 states that the total number of nests in the cycle is less than or equal to the length

of the basic chain. Further, each nest contains at most one H-relevant pole by Lemma 2.2. This gives that the number of H-relevant poles is at most the number of nests. Hence the number of H-relevant poles is at most the length of the basic chain. $\square$

Remark 3.1. *Let h, n and l denote the number of H-relevant poles, the number of nests and the length of the basic chain corresponding to a cycle of Herman rings respectively. Then it follows from the proof of the above lemma that $h \leq n \leq l$. If $h = l$ then $h = n = l$. It is evident from the proof of Lemma 3.1 that each nest contains at least one basic ring. Therefore, each nest contains exactly one basic ring in this case. Also each nest contains an H-relevant pole. As evident from Lemma* 3.1 *of [3], this is the case if the period of the Herman ring is three.*

Remark 3.2. *If there is a* 4*-periodic Herman ring, it is seen (Lemma 3.2, [3]) that the length of the basic chain is always three whereas the number of H-relevant poles is always two. Further, the number of nests can be two or three.*

Note that two different chains do not contain a common ring. As will be evident from Remark 3.4, each ring of a cycle that surrounds O_f as well as a pole does not belong to any chain. But all other rings are in some chain. Since the number of all rings in a cycle of Herman rings is the period of the cycle, the lengths of chains are crucial. Next result determines the number of independent chains and their lengths in terms of the number of H-relevant poles.

Theorem 3.3. *Let H be a p-periodic Herman ring of f and h be the number of H-relevant poles. Then the number of independent chains is $h-1$ or h. It is h whenever the basic nest does not surround any pole. If $c \in \{h-1, h\}$, and $C_2, C_3, \ldots, C_{c+1}$ are the independent chains such that $|C_2| < |C_3| < \cdots < |C_{c+1}|$ then $|C_j| \geq j$ for all j, $2 \leq j \leq c+1$.*

Proof. Let N be a non-basic nest surrounding a pole. This means that a ring, say H_r belonging to N surrounds a pole w of f. Such a nest exists as there are at least two H-relevant poles. The ring H_{r-1}, the periodic pre-image of H_r does not surround any pole by Remark 2.6. Since H is periodic, there is an m such that H_{r-m}, the periodic pre-image of H_r under f^m, surrounds a pole. Choose the smallest such m and observe that $m \geq 2$. Then H_{r-m+1} does not surround any pole. Further, the ring H_{r-m+1} is in the basic nest by Remark 2.6. Thus $\{H_{r-m+1}, H_{r-m+2}, \dots, H_r\}$ is a chain. Therefore, for each non-basic nest surrounding a pole w, there is a chain corresponding to w.

Each H-relevant pole, except possibly one is surrounded by a non-basic nest. In other words, the number of non-basic nests surrounding some pole is either $h-1$ or h. It follows from the previous paragraph that the number of independent chains is either $h-1$ or h. If the basic nest does not surround any pole then the number of different non-basic nests surrounding some pole is h. This is nothing but the number of independent chains.

Since lengths of two different independent chains are different (Lemma 2.3), the chains can be ordered according to their lengths. Let $|C_2| < |C_3| < \cdots < |C_{c+1}|$.

Suppose that $|C_j| < j$ for some j, $2 \leq j \leq c+1$. Note that the length of each chain is at least two. In particular $|C_2| \geq 2$, which gives that $2 \leq |C_2| < |C_3| < \cdots < |C_j| \leq j-1$. However, this is not possible by the Pegionhole Principle unless two chains have same length. However, this is not true. Therefore, $|C_j| \geq j$, for all $j = 2\ , 3, \dots,\ c+1$.

□

Following remark deals with the situation when the basic nest surrounds a pole.

Remark 3.4. *If the basic nest surrounds a pole w then it is still possible to have exactly $h-1$ independent chains. In fact, this is true when each*

ring surrounding w also surrounds O_f. To prove it, note that there are $h - 1$ independent chains corresponding to each H-relevant pole surrounded by a non-basic nest (it follows from Theorem 3.3). In order to show that those are the only independent chains, let there be a chain $\{H_r, H_{r+1}, \ldots, H_{k-1}, H_k\}$ corresponding to w. Then H_k surrounds w as well as O_f. Since H_{k-1} does not surround any pole (by definition of chain), $f : B(H_{k-1}) \to B(H_k)$ is conformal. This is a contradiction as $B(H_k)$ contains O_f and each component of the pre-image of $B(H_k)$ is unbounded (See Lemma 2.1, [5]). Hence there is no chain corresponding to w and the number of independent chains is exactly $h - 1$ whenever each ring surrounding w also surrounds O_f.

We now prove the first main result of this article.

Theorem 3.5. (Lower bound for the period of Herman ring) *Let H be a p-periodic Herman ring of a function f and h be the number of H-relevant poles.*

1. *If the basic nest surrounds a pole then $p \geq \frac{h(h+1)}{2}$.*
2. *If the basic nest does not surround any pole then $p \geq \frac{h(h+3)}{2}$.*

Proof. Note that two rings belonging to two different independent chains are different.

1. If the basic nest surrounds a pole w then the number of independent chains is $h-1$ or h. If it is $h-1$ then each independent chain corresponds to a pole surrounded by a non-basic nest. The total number of rings contained in all these independent chains is $|C_2| + |C_3| + \cdots + |C_h|$. This value is at least $2 + 3 + \cdots + h = \frac{h(h+1)}{2} - 1$ by Theorem 3.3. Further, there is a ring in the basic nest surrounding w and not belonging to any independent chain. Therefore, the number of rings in the cycle of H is at least $\frac{h(h+1)}{2}$. In other words, $p \geq \frac{h(h+1)}{2}$.

If the number of independent chains is h then the total number of rings contained in all these chains is $|C_2| + |C_3| + \cdots + |C_{h+1}|$. This number is at least $2+3+\cdots+(h+1) = \frac{(h+1)(h+2)}{2} - 1 = \frac{h(h+3)}{2}$ by Theorem 3.3. Therefore, $p \geq \frac{h(h+3)}{2}$ which is clearly bigger than $\frac{h(h+1)}{2}$.

2. If the basic nest does not surround any pole, then there exists h number of independent chains. Arguing as in the previous paragraph, it follows that the total number of rings in the cycle of H is greater than or equal to $|C_2| + |C_3| + \cdots + |C_{h+1}| \geq 2+3+\cdots+(h+1) = \frac{h(h+3)}{2}$. Thus $p \geq \frac{h(h+3)}{2}$.

$\square$

An additional assumption leads to equality in Theorem 3.5. The assumption is satisfied for 3-periodic Herman rings [3]. Recall that h denotes the number of H-relevant poles.

Theorem 3.6. *Let H be a p-periodic Herman ring and each ring surrounding an H-relevant pole be the outermost ring of the concerned nest.*

1. *If the length of the basic chain is equal to h then $p = \frac{h(h+1)}{2}$.*

2. *If the length of the basic chain is equal to $h+1$ and the basic nest does not surround any pole then $p = \frac{h(h+3)}{2}$.*

Proof. We assert that the set of all chains is the same as the set of all independent chains. Equivalently, two different chains are independent. To prove this by contradiction, suppose that $\{H_r,\ H_{r+1}, \cdots,\ H_k\}$ and $\{H_i,\ H_{i+1}, \cdots,\ H_{k'}\}$ are two different chains and are not independent. Then they correspond to the same pole. In other words, H_k and $H_{k'}$ are different rings surrounding the same pole. However, this negates our assumption that only the outermost ring of a nest surrounds a pole. The period of H is now to be determined by finding the lengths of all independent chains.

1. Since the length of the basic chain is equal to h, there are exactly h number of nests and each nest contains exactly one basic ring. Further, each nest surrounds an H-relevant pole. All these statements are observed in Remark 3.1. In particular, the basic nest surrounds a pole. Since the length of the basic chain is h, the number of independent chains is $h-1$ or h, by Theorem 3.3. In fact, this number is $h-1$ by Remark 3.4. Let the $h-1$ chains be denoted by $C_2,\ C_3,\ \ldots,\ C_h$. Then $2 \leq |C_2| < |C_3| < \cdots < |C_h|$ by Theorem 3.3. Further, $|C_h| = h$ by assumption. This is possible only when $|C_j| = j$ for all $j = 2, 3, \ldots, h$. Thus the total number of rings in the cycle of H is $1+2+\cdots+h = \frac{h(h+1)}{2}$. Note that the first term 1 in the sum corresponds to the outermost ring of the basic nest. Thus $p = \frac{h(h+1)}{2}$.

2. It follows from Theorem 3.3 that the number of independent chains is h. Let the independent chains be denoted by $C_2,\ C_3,\ \ldots,\ C_{h+1}$. Then $2 \leq |C_2| < |C_3| < \cdots < |C_{h+1}|$ by Theorem 3.3. Further, $|C_{h+1}| = h+1$ by assumption. This is possible only when $|C_j| = j$ for all $j = 2, 3, \ldots, h+1$, again by the Pegionhole Principle. Thus the total number of rings in the cycle of H is $2+3+\cdots+(h+1) = \frac{h(h+3)}{2}$ proving that $p = \frac{h(h+3)}{2}$.

$\square$

Remark 3.7. *The assumption in Theorem 3.6(1) gives that the basic nest surrounds a pole. This follows from the proof.*

Theorem 3.5 gives a lower bound for the period of a Herman ring (if exists) in terms of the number of H-relevant poles. The following theorem provides a lower bound for the period of a periodic Fatou component containing an omitted value in terms of the number of H-relevant poles.

Theorem 3.8. *Let U be a periodic Fatou component of f containing at least one omitted value. Also let H be a Herman ring of f. Then the number of*

H-relevant poles is strictly less than the period of U. In particular, if U is invariant or 2-periodic then f has no Herman ring.

Proof. Let U be a q-periodic Fatou component containing an omitted value of f and $\{U = U_1,\ U_2, \ldots, U_q\}$ be the cycle. Note that a Herman ring cannot contain any omitted value and hence U is not a Herman ring. Also let h be the number of H-relevant poles and $\{H_1, H_2, \ldots, H_l\}$ be the basic chain. Since H_1 surrounds O_f, it surrounds U_1. This implies that H_i surrounds U_i for $i = 1,\ 2, \ldots,\ l$ by the Maximum Modulus Principle. Further, all such $U_i's$ are bounded. But U_q is unbounded by Lemma 2.1, [5]. Therefore $q > l$. Note that $l \geqslant h$ by Lemma 3.2. Thus $h < q$. In other words, the number of H-relevant poles is strictly less than the period of U.

If U is invariant or 2-periodic then $q = 1$ or 2 and consequently $h < 2$ which is not possible as number of H-relevant poles is at least 2 ([3]). Thus f has no Herman ring if U is invariant or 2-periodic. □

Remark 3.9. *Examples of functions in M_o with periodic Fatou components containing an omitted value can be found in [8, 9].*

Remark 3.10. *If U is a 3-periodic Fatou component of f containing an omitted value and H is a Herman ring of f, then the number of H-relevant poles is two by Theorem 3.8. It follows from the proof that H_1 surrounds U. Since one of U, U_1, U_2 is unbounded, the number of rings in the basic chain is at most two. In fact, it is exactly two. Now it follows from Lemma 3.1 that there are only two nests. Further, each nest surrounds exactly one H-relevant pole. For two Herman rings H, H' belonging to two different cycles, the set of H-relevant poles coincides with the set of H'-relevant poles.*

Remark 3.11. *It is shown in [3] that the length of the basic chain is three for every 4-periodic Herman ring. It follows from the previous remark that, if f has a 3-periodic Fatou component containing an omitted value then it has no 4-periodic Herman ring.*

References

[1] K. Barański, N. Fagella, X. Jarque, and K. Bogusława, *Connectivity of Julia sets of Newton maps: a unified approach*, Rev. Mat. Iberoam. 34 (2018), no. 3, 1211–1228.

[2] W. Bergweiler, *Iteration of meromorphic functions*, Bull. Amer. Math. Soc. (N.S.), 29 (1993), no. 2, 151-188.

[3] T. Chakra, T. Nayak, and G. Chakraborty, *Herman rings with small periods and omitted values*, Acta Math. Sci. Ser. B (Engl. Ed.) 38 (2018), no. 6, 1951–1965.

[4] P. Dominguez and N. Fagella, *Existence of Herman rings for meromorphic functions*,Complex Var. Theory Appl., 49 (2004), no. 12, 851-870.

[5] T. Nayak, *On Fatou components and omitted values*, Geometry, groups and dynamics, Amer. Math. Soc., 639 (2015), 349–358.

[6] T. Nayak, *Herman rings of meromorphic maps with omitted value*, Proc. Amer. Math. Soc. 144 (2016), 587-597.

[7] T. Nayak and J. H. Zheng, *Omitted values and dynamics of transcendental meromorphic functions*, J. Lond. Math. Soc., 83 (2011), no. 1, 121-136.

[8] T. Nayak and M. G. P. Prasad, *Iteration of certain meromorphic functions with unbounded singular values*, Ergod. Theory Dyn. Syst. 30 (2010), 877–891.

[9] M. Guru Prem Prasad and T. Nayak, *Dynamics of* $\lambda \tanh(e^z)$, Discrete and Continuous Dynamical Systems - Series A 19, (September 2007), no. 1, 121–138.

[10] J. H. Zheng, *Remarks on Herman rings of transcendental meromorphic functions*, Indian J. Pure Appl. Math. 31 (2000), no. 7, 747-751.